Fibonacci Numbers

Recall the definition of the Fibonacci numbers.

Definition. The *Fibonacci sequence* is the sequence of numbers $F_0, F_1, F_2, \cdots$ defined recursively by the conditions $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$.

Here is a table of the first $1 + 20$ Fibonacci numbers:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

There are many interesting relationships among Fibonacci Numbers. We list some of these now.

1. Show $\displaystyle\sum_{k=1}^{n} F_k = F_{n+1} - 1$ for all integers $n \geq 1$.
2. Show $\displaystyle\sum_{k=1}^{n} F_{2k-1} = F_{2n}$ for all integers $n \geq 1$.
3. Show $\displaystyle\sum_{k=0}^{2n} (-1)^k F_k = F_{2n-1} - 1$ for all integers $n \geq 1$.
4. Show $\displaystyle\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$ for all integers $n \geq 1$.
5. Show $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for all integers $n \geq 1$.
6. Show $F_{2n+5} = 3F_{2n+3} - F_{2n+1}$ for all integers $n \geq 0$.[1]
7. Show $F_{2n+4} = 3F_{2n+2} - F_{2n}$ for all integers $n \geq 0$.[2]
8. Show $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all integers $n \geq 1$.[3]
9. Show F_{3n} is divisible by 2 for all integers $n \geq 0$.
10. Show F_{5n} is divisible by 5 for all integers $n \geq 0$.
11. Suppose $S_0, S_1, S_2, \cdots$ is an arbitrary sequence satisfying the recursion $S_n = S_{n-1} + S_{n-2}$ for all $n \geq 2$.
    (a) Show $S_n = S_0 F_{n-1} + S_1 F_n$ for all integers $n \geq 1$.
    (b) Show $F_{a+b+1} = F_a F_b + F_{a+1} F_{b+1}$ for all $a, b \in \mathbb{N}_0$.[4]
    (c) Show F_n divides F_{nk} for all $n \in \mathbb{N}_0$ and all $k \in \mathbb{N}_0$.[5]
    (d) Show $F_{a+r}F_{b+r} - F_{c+r}F_{d+r} = (-1)^r (F_a F_b - F_c F_d)$ for all $a, b, c, d, r \in \mathbb{N}_0$ with $a + b = c + d$.[6]
12. Show $F_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$ for all integers $n \geq 1$.
13. Prove that for all $n \in \mathbb{N}_0$ there exist unique $s, t \in \mathbb{N}_0$ such that the following conditions all hold:
    (a) $n = F_s + F_t$
    (b) $|s - t| > 1$
    (c) $s \neq 2 \neq t$

[1] HINT: You don't need induction–the Fibonacci recursion is enough.
[2] HINT: The same method from the previous problem will work here...
[3] HINT: Use induction and the preceding two identities to prove this.
[4] HINT: Make a related sequence and use part a.
[5] HINT: Use part b. You can use induction either on n or on k–it's up to you to figure out which one makes sense.
[6] HINT: Induct on $r \in \mathbb{N}_0$ using part b for the inductive step.