Matchings in Simple Graphs

All graphs in this document are assumed to be simple and have finitely many vertices.

Basic Definitions and Examples

Recall that for all graphs G and all $S \subseteq V(G)$, the *neighborhood* of S in G is

$$N_G(S) \coloneqq \{v \in V(G) : vs \in E(G) \text{ for some } s \in S\}.$$

Definition. Let G be a simple graph. A *(partial) matching* on G is a set $M \subseteq E(G)$ such that for all $e, e' \in M$ we have $e \cap e' \neq \emptyset$ implies $e = e'$.

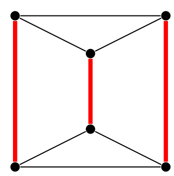
Intuitively, a partial matching in a graph is a set of edges which do not share any endpoints.

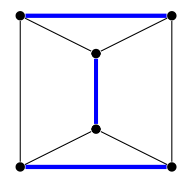
Example. The empty set $M = \emptyset$ is a partial matching on every graph.

The previous example shows that every graph has a partial matching.

Definition. Let G be a simple graph. A *perfect matching* on G is a partial matching $M \subseteq E(G)$ such that for all $v \in V(G)$ there is an $e \in M$ such that v is an endpoint of e.

Example. The red edges (respectively blue edges) in the graph G below form a perfect matching.





Problem. Let $n \in \mathbb{Z}_{>0}$ be an even number.

1. How many perfect matchings does the complete graph K_n have?
2. How many perfect matchings does the cycle graph C_n have?
3. How many perfect matchings does the path graph P_n have?

Problem. How many perfect matchings does the Petersen graph have?[1]

The following is a very natural observation.

Proposition. *If G is a simple graph with a perfect matching, then G has an even number of vertices.*

Proof. Homework. □

Hall's Marriage Theorem

How do we know when a graph G has a perfect matching? In general, this is a rather difficult question. For certain special types of graphs, this question can be answered more easily than others. Recall the following definition.

Definition. A graph G is *bipartite* when we can write $V(G) = L \cup R$ for disjoint sets $L, R \subseteq V(G)$ such that every edge $e \in E(G)$ has one end in L and the other end in R. We call $L \cup R$ a *bipartiton* of G.

Example. Let $m, n \in \mathbb{N}_0$. The *complete bipartite graph* $K_{m,n}$ is given by

$$V(K_{m,n}) \coloneqq \{1\} \times [m] \cup \{2\} \times [n] \qquad \text{and} \qquad E(K_{m,n}) \coloneqq \{\{(1,i),(2,j)\} : i \in [m] \text{ and } j \in [n]\}.$$

Problem. How many perfect matchings does the complete bipartite graph $K_{n,n}$ have?

For the class of bipartite graphs we can get a complete answer to the question. We will say that a matching M *covers* a set $S \subseteq V(G)$ when every $s \in S$ is the endpoint of some edge in M.

Proposition (Hall's Marriage Theorem)**.** *Let G be a bipartite graph on bipartition $V(G) = L \cup R$. There is a partial matching in G covering L if and only if $\#S \leq \#N_G(S)$ holds for all $S \subseteq L$.*

[1] **HINT**: Use cases and the symmetry of the Petersen graph.

Proof. Let G be a bipartite graph on bipartiton $V(G) = L \cup R$.

($\Rightarrow$): Suppose G has a partial matching M covering L, and let $S \subseteq L$ be arbitrary. Because M covers L, for all $s \in S$ there is a unique $t_s \in V(G)$ such that $st_s \in M$. Moreover, $t_s = t_{s'}$ implies $s = s'$ because M is a partial matching; thus $\#S = \#\{t_s : s \in S\}$. Moreover $\{t_s : s \in S\} \subseteq N_G(S)$ by definition; hence $\#S \leq \#N_G(S)$ as desired.

($\Leftarrow$): Assume $\#S \leq \#N_G(S)$ holds for all $S \subseteq L$. We proceed by strong induction on $n = \#L$.

Base Cases: If $\#L = 0$, then $M = \emptyset$ is a partial matching covering $L = \emptyset$. If $\#L = 1$, then $L = \{v\}$ for some $v \in V(G)$; moreover $\#N_G(L) \geq 1$ yields an element $u \in N_G(v)$, so $M = \{uv\}$ is a partial matching covering L.

Inductive Step: Suppose the result holds for all graphs with fewer than n vertices in L. Either $\#S < \#N_G(S)$ for all $\emptyset \subsetneq S \subsetneq L$ or there is a subset $\emptyset \subsetneq S \subsetneq L$ with $\#S = \#N_G(S)$.

Case 1: If $\#S < \#N_G(S)$ for all $\emptyset \subsetneq S \subsetneq L$, then choose any $v \in L$ and note that $1 = \#\{v\} \leq N_G(v)$ allows us to choose some $u \in N_G(v)$; define a new graph G' with $V(G') \coloneqq V(G) \setminus \{u, v\}$ and $E(G') = \{e \in E(G) : u, v \notin e\}$. Note that G' is bipartite with bipartition $L' \coloneqq L \setminus \{v\}$ and $R' \coloneqq R \setminus \{u\}$. Now for all $S \subseteq L'$, we have $N_{G'}(S) = N_G(S) \setminus \{u\}$; thus $\#S < \#N_G(S)$ implies $\#S \leq \#N_G(S) - 1 \leq \#(N_G(S) \setminus \{u\}) = \#N_{G'}(S)$. Hence by induction G' has a perfect matching M'; letting $M \coloneqq M' \cup \{uv\}$ we have that M is a perfect matching on G.

Case 2: If $\#X = \#N_G(X)$ for some $\emptyset \subsetneq X \subsetneq L$, we define two new graphs H and K by

$$V(H) = X \cup N_G(X) \qquad E(H) = \{uv \in E(G) : u, v \in X \cup N_G(X)\}$$
$$V(K) = V(G) \setminus (X \cup N_G(X)) \qquad E(K) = \{uv \in E(G) : u, v \notin X \cup N_G(X)\}.$$

Note that $V(H) = X \cup N_G(X)$ and $V(K) = (L \setminus X) \cup (R \setminus N_G(X))$, and both H and K are bipartite because G is bipartite on $L \cup R$. Moreover $\#X < n$ and $\#(L \setminus X) < n$. For all $S \subseteq V(H)$ we have $\#N_H(S) = \#N_G(S) \geq \#S$ as $N_H(S) = N_G(S)$; thus H has a perfect matching M_H by the inductive hypothesis. Assume to the contrary $\#S > \#N_K(S)$ for some $S \subseteq V(K)$; now $N_G(S \cup X) = N_G(S) \cup N_G(X) = N_K(S) \cup N_G(X)$, so we obtain

$$\#(S \cup X) = \#S + \#X > \#N_K(S) + \#X = \#N_K(S) + \#N_G(X) = \#N_G(S \cup S),$$

which is absurd by our assumption on G. Hence $\#S \leq \#N_K(S)$ for all $S \subseteq V(K)$ yields that K has a perfect matching M_K by the inductive hypothesis. Hence $M \coloneqq M_H \cup M_K$ is a perfect matching on G.

We conclude that the original statement is true! □

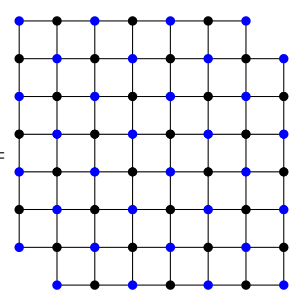
The following yields a complete characterization of bipartite graphs with a perfect matching.

Corollary. *A simple bipartite graph G has a perfect matching if and only if there is a bipartition $V(G) = L \cup R$ with $\#L = \#R$ and $\#S \leq \#N_G(S)$ for all $S \subseteq L$.*

Proof. Exercise! □

Hall's Marriage Theorem is very useful for showing that a bipartite graph G does not have a perfect matching.

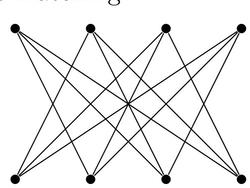
Example. Does the following bipartite graph have a perfect matching?

$G =$ 

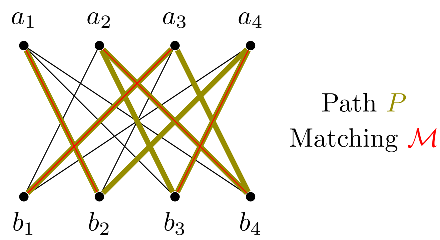
Solution: Graph G has bipartition $A \sqcup B$ with $\#A = 32$ and $\#B = 30$; thus G does not admit a matching.

At the moment, we can only observe perfect matchings.

Example. Does the following graph have a perfect matching?

$G =$ 

Solution: Consider path $P = (a_1, b_2, a_4, b_3, a_2, b_4, a_3, b_1)$ in G and the corresponding matching $\mathcal{M}_P$:



One observes that $\mathcal{M}$ is a perfect matching in G, as desired.

Algorithms to Compute Perfect Matchings in Bipartite Graphs

Hall's Marriage Theorem is a complete characterization of the bipartite graphs which admit perfect matchings. However Hall's Marriage Theorem is not practical for deciding whether a bipartite graph has a perfect matching; it is useful to verify that a bipartite graph does not have a perfect matching–verifying $\#S > \#N_G(S)$ (and thus showing no perfect matching exists) is straightforward. Using the condition to verify that a graph has a perfect matching requires checking $2^{\#L} - 1$ separate inequalities (one for each nonempty subset of L).

Our next goal is to describe an algorithm which either produces a perfect matching in G or produces an example of a set $S \subseteq L$ with $\#S > \#N_G(S)$. This is the best type of algorithm we can hope for; it both answers the question "does G have a perfect matching?" and produces a *certificate* which can be used to quickly verify the answer.

Let G be a simple graph with a partial matching M. An *augmenting path* is a path P which alternates edges in M and edges not in M and so that both endpoints of P are not covered by M.

Algorithm (Augmenting Paths)**.** Let G be a bipartite graph with bipartition $V(G) = L \sqcup R$ and let $M_0 = \emptyset$.

1. Having a partial matching M_i, check if M_i covers L.
   (a) If M_i covers L, output M_i and stop.
   (b) If M_i does not cover L, choose an unmatched vertex $v_i \in L$.
      i. If there is an augmenting path P_i from v_i to an unmatched vertex, define $M_{i+1} := M_i \triangle E(P_i)$.
      ii. If there is no augmenting path from v_i to another unmatched vertex, choose a maximal path Q from v_i which alternates edges in M_i and edges not in M_i. Output the pair $(L \cap V(Q), V(Q) \setminus L)$ and stop.
   (c) Increment i and return to the beginning of the algorithm.

Motivated readers are encouraged to try their hand at the following project (and to think about how fast this algorithm is by comparison to the naïve one described before stating the algorithm).

Project. [2] Implement the Augmenting Paths Algorithm in your favorite programming language.

The following proposition (and its corollary) show that the Augmenting Paths Algorithm accomplishes our goal; the algorithm always outputs an object which can be quickly checked to decide whether or not the bipartite graph G has a partial matching covering L.

Proposition. *Consider the Augmenting Paths Algorithm (APA) on a bipartite graph G with bipartition $L \cup R$.*

1. *The APA terminates after finitely many steps.*
2. *If a set M is the output of the APA, then M is a partial matching covering L in G.*
3. *If a pair (S, T) is the output of the APA, then $S \subseteq L$ has $N_G(S) = T$ and $\#S = \#T + 1$.*

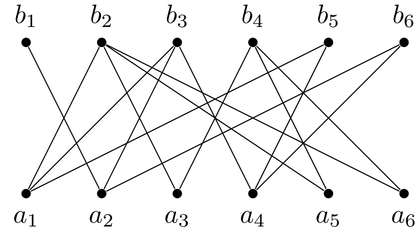
The proof is not hard, but does take time; you are encouraged to write a proof on your own.

Corollary. *A bipartite graph G has a partial matching covering L if and only if the APA returns one.*

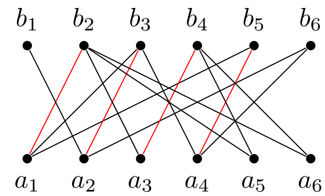
[2] This is not required, but I would be quite impressed with your initiative if you choose to do so.

We will now illustrate the Augmenting Paths Algorithm with an example.

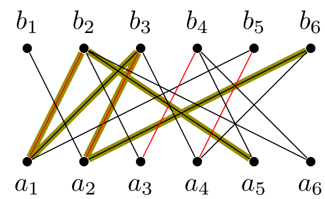
Example. Does the following bipartite graph have a perfect matching?

$G =$ 

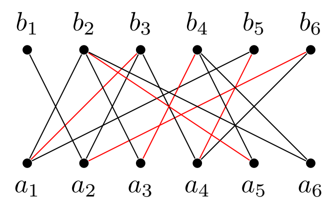
Solution: First we greedily choose the partial matching $\mathcal{M}_0 = \{a_1b_2, a_2b_3, a_3b_4, a_4b_5\}$; note that $\mathcal{M}_0$ is maximal because $N_G\{a_5, a_6\} \subseteq \{b_2, b_3, b_4, b_5\}$. We have $U_0 = \{a_5, a_6, b_1, b_6\}$ for this matching.



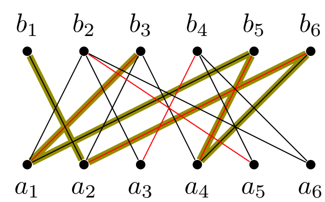
Now we can find an augmenting path $P_0 = (a_5, b_2, a_1, b_3, a_2, b_6)$ in $(G, \mathcal{M}_0)$.



Now we obtain the new partial matching $\mathcal{M}_1 = \mathcal{M}_0 \triangle E(P_0)$ with unmatched vertices $U_1 = \{a_6, b_1\}$.



Now note that any augmenting paths must start at b_1 and end at a_6. But the maximal alternating path starting at b_1 is uniquely determined to be $P = (b_1, a_2, b_6, a_4, b_5, a_1, b_3)$, and this path also ends in B.



Now for this path, notice that we have the following:

$$\#\{b_1, b_3, b_5, b_6\} = 4 > 3 = \#\{a_1, a_2, a_4\} = \#N_G\{b_1, b_3, b_5, b_6\}$$

Hence G does not have a perfect matching by the Marriage Theorem.