Group Quiz

Instructions: Complete each of the following on separate, stapled sheets of paper.

1. Prove that the graph $K_{m,n}$ has mn edges.

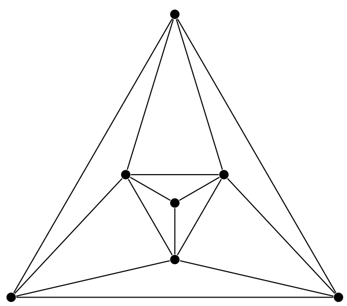
Solution: Recall that $V(K_{m,n}) = (\{1\}\times[m])\cup(\{2\}\times[n])$ and $E(K_{m,n}) = \{\{(1,i),(2,j)\} : i \in [m] \text{ and } j \in [n]\}$. Define a function $f : [m] \times [n] \to E(K_{m,n}) : (i,j) \mapsto \{(1,i),(2,j)\}$; it is easy to show that f is bijective. Hence $\#E(K_{m,n}) = mn$ as desired.

2. What is the smallest number of edges that must be removed from K_5 to make a bipartite graph?

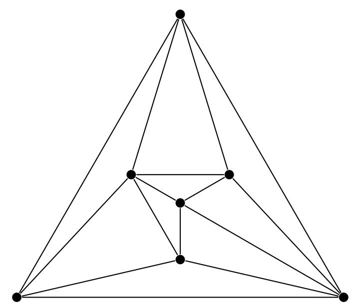
Solution: If $G \leq K_5$ is bipartite on vertex bipartition $V(G) = L \cup R$, then $\#L + \#R = 5$; moreover the graph G is complete bipartite if G maximizes the number of edges. Up to isomorphism $G = K_{0,5}$, $G = K_{1,4}$, or $G = K_{2,3}$. Among these $K_{2,3}$ has the most edges; as K_5 has 10 edges, the desired quantity is $10 - 6 = 4$.

3. For each of the graphs G below, compute the chromatic number $\chi(G)$. Give a complete proof.

(a)



(b)



4. Prove that every finite simple graph G has at least $\binom{\chi(G)}{2}$ edges (where $\chi(G)$ is the chromatic number of G).

Solution: Let G be a finite simple graph, and let c be a coloring of G by $\chi(G)$ colors. Assume to the contrary that there are two colors a and b so that if $u, v \in V(G)$ have $c(u) = a$ and $c(v) = b$, then $uv \notin E(G)$. Now build a new coloring c' by

$$c'(x) = \begin{cases} c(x) & \text{if } c(x) \neq b \\ a & \text{if } c(x) = b \end{cases}.$$

By our assumption c' is a proper coloring of G using 1 color fewer; but c used $\chi(G)$ colors, which implies the absurdity $\chi(G) < \chi(G)$. Hence for each pair $\{a,b\}$ of colors in a $\chi(G)$-coloring of G there is an edge $e \in E(G)$ connecting an a-colored vertex to a b-colored vertex. Hence $\#E(G) \geq \binom{\chi(G)}{2}$ as desired.

5. Let G be a graph and let $\sim$ be the relation on $V(G)$ defined by $u \sim v$ when there is a walk in G from u to v. Prove that $\sim$ is an equivalence relation.

Solution: Let G be an arbitrary graph and consider $\sim$ defined as above. Let $u, v, w \in V(G)$ be arbitrary.

Reflexive: Because $W = (u)$ is a walk starting and ending at u, we have that $u \sim u$.

Symmetric: Assume $u \sim v$. Thus there is a walk $W = (w_0, w_1, \cdots, w_k)$ in G with $w_0 = u$ and $w_n = v$. Reverse this walk to obtain walk $\bar{W} = (w_k, w_{k-1}, \cdots, w_0)$ in G starting at v and ending at u. Hence $v \sim u$.

Transitive: Assume $u \sim v$ and $v \sim w$. Thus there are walks $A = (a_0, a_1, \cdots, a_k)$ and $B = (b_0, b_1, \cdots, b_m)$ in G with $a_0 = u$, $a_k = v$, $b_0 = v$, and $b_m = w$. Now concatenating these paths we obtain a new path $AB = (u = a_0, a_1, \cdots, a_k = b_0, b_1, \cdots, b_m = w)$ starting at u and ending at w.